UNIFICATION OF THE λ-POINT *A*-ARY SUBDIVISION SCHEME

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ABSTRACT*—We propose and analyze a subdivision scheme, which generates the mask of all stationary approximating subdivision schemes in its compact form and produces complex geometrical structures with higher smoothness. The performance of the new schemes is demonstrated by several examples. Moreover, all B-splines and many other well-known schemes [1, 11, 13, 15, 16] are special cases of our proposed scheme.*

Keywords: Approximating subdivision scheme; *a*-ary schemes; continuity and Laurent polynomial
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I. INTRODUCTION

In recent years, the subject of subdivision gained popularity due to some new applications, such as 3D computer graphics, and due to close relation of subdivision analysis to wavelet analysis. Subdivision algorithms are most suitable for computer applications; they are simple to apprehend, easy to implement, highly flexible and very attractive to the users and researchers. In free form surface design applications, such as in the 3D animation industry, subdivision methods are already in extensive use, and the next venture is to introduce these methods to more conservative and demanding to the world of geometric modeling in the industry.

Rham [1] and Chaikin [2] are regarded as the pioneers in the field of subdivision. Although they developed the corner cutting schemes, but important steps in the sub-division schemes have been made in the last two decades, and the subject expanded in new directions due to various generalizations and applications. The idea of families of subdivision schemes of higher arity is relatively new. Based on wavelet theory, Lian [2] introduced 2*m*-point *a*-ary for any $a \geq 2$ and $(2m + 1)$-point *a*-ary for any odd $a \geq 3$ interpolatory subdivision schemes for curve design. These schemes include the extended family of the classical 4- and 6-point [3] and the family of the 3- and 5-point *a*-ary interpolatory schemes [4]. Zheng *et al*. [5] investigated ternary interpo-latory schemes with an odd number of control points, namely, $(2n - 1)$-point ternary interpolatory subdivision schemes. They also investigated ternary even symmetric *p*-ary [6] and 2*n*-point [7] approximating subdivision schemes. Mustafa and Najma [8] presented general formulae for the mask of $(2b + 4)$-point *n*-ary approximating as well as interpolating subdivision schemes for any integers $b \geq 0$ and $n \geq 2$. These formulae

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subdivision schemes and also given derivation of some family members.

For the analysis of binary, ternary and quaternary schemes, we may refer to [10], [11] and [12]. Analysis of higher arity schemes can be performed in a similar fashion. Main objective of the current paper is to introduce λ-point *a*-ary non-parametric as well as parametric approximating subdivision schemes for curve design for any integers*;* $a \geq 2$, which unifies all the approximating subdivision schemes. This subdivision also provides variety of even-point and odd-point even-ary and odd-ary approximating parametric and non-parametric schemes generated by an explicit formulae in a single platform with high continuity than existing schemes generated by an explicit formulae.

2 Analysis of the general *a*-ary -point approximating scheme.

A general compact form of univariate *a*-ary subdivision scheme S which maps a polygon $f^k = \{f_i^k\}_{i\in\mathbf{Z}}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbf{Z}}$ is defined by

$$f_i^{k+1} = \sum_{j\in Z} \alpha_{aj-i} f_j^k \quad i \in \mathbf{Z} \tag{2.1}$$

where the set = $\{a_i : i \in \mathbf{Z}\}$ of coefficients is called the mask at *k*-th level of refinement. A necessary condition for the uniform convergence of subdivision scheme (2.1) is that

$$\sum_{j\in\mathbf{Z}} \alpha_{aj} = \sum_{j\in\mathbf{Z}} \alpha_{aj+1} = \sum_{j\in\mathbf{Z}} \alpha_{aj+2} = \ldots\ldots\ldots = \sum_{j\in\mathbf{Z}} \alpha_{aj+a-1} = 1 \tag{2.2}$$

A subdivision scheme is uniformly convergent if for any initial data $f^0 = \{f_i^0 : i \in \mathbf{Z}\}$, there exists a continuous function *f* such that for any closed interval $I \subset$R, it satisfies

$$\lim_{x\to\infty} \sup_{i\in a^k 1} |f_i^k - f(a^{-k} i)| = 0$$

Obviously, $f = S^\infty f^0$

Introducing a symbol called Laurent polynomial

$$\alpha(z) = \sum_{i\in Z} \alpha_i z^i \tag{2.3}$$

of the mask α= { α_i : $i \in \mathbf{Z}$ } which play an efficient role to analyze the convergence and smoothness of subdivision scheme. From (2.2) and (2.3) the Laurent polynomial of convergent subdivision scheme satisfies.

$$\alpha(e^{4ih\pi/a}) = 0.\ h \in Z \cap (0,a) \quad and \quad \alpha(1) = a \tag{2.4}$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of an associated Laurent polynomial $\alpha^{(1)}(z)$

$$\alpha^{(1)}(Z) = az^{z-1}\left(\frac{1-z}{1-z^a}\right)\alpha(z)$$

The subdivision scheme S_1 with Laurent polynomial $\alpha^{(1)}(z)$, is related to the scheme S with Laurent polynomial $\alpha(z)$ by the following theorem.

Theorem 2.1. *[11] Let S denote a subdivision scheme with Laurent polynomial $\alpha(z)$ satisfying* (2.4). *Then there exists a subdivision scheme S_1 with the property.*

$$\Delta f^k = S_1 \Delta f^{k-1}$$

where $f^k = S^k f^0$ and $\Delta f^k = \{(\Delta f^k)_i = a^k(f^k_{i+1} - f^k_i); i \in \mathbf{Z}\}$.

Furthermore, S is a uniformly convergent if and only if $\frac{1}{a}S_1$ converges uniformly to zero function for all initial data f^0, in the sense that

$$\lim_{k\to\infty}\left(\frac{1}{a}S_1\right)^k f^0 = 0.$$

The above theorem indicates that for any given scheme S, with the mask α satisfying (2.2), we can prove the uniform convergence of S by deriving the mask of $\frac{1}{a}S_1$ and computing $\left\|\left(\frac{1}{a}S_1\right)^i\right\|_\infty$ for $i = 1,2,3........,L,$ where L is the first integer for which $\left\|\left(\frac{1}{a}S_1\right)^L\right\|_\infty < 1$. If such an L exists, then S coverage's uniformly. Since there are a rules for computing the values at the next refinement level, so we define the norm

$$\|S\|_\infty = \max\left\{\sum_{j\in\mathbf{Z}}|\alpha_{aj}|, \sum_{j\in\mathbf{Z}}|\alpha_{aj+2}|, \sum_{j\in\mathbf{Z}}|\alpha_{aj+2}|, ...\sum_{j\in\mathbf{Z}}|\alpha_{aj+a-1}|\right\}, \quad (2.5)$$

and

$$\left\|\left(\frac{1}{a}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in z}|b^{n,L}_{i+a^L j}|; i = 0,1,2,......,a^L - 1\right\} \quad (2.6)$$

where

$$b^{[n,L]}(z) = \frac{1}{a^L}\prod_{j=0}^{L-1}\alpha^{(n)}(z^{a^j}) \quad (2.7)$$

and

$$\alpha^{(n)}(z) = \left(az^{a-1}\left(\frac{1-z}{1-z^a}\right)\right)\alpha^{(n-1)}(z) = \left(az^{a-1}\left(\frac{1-z}{1-z^a}\right)\right)^n \alpha(z), n \geq 1. \quad (2.8)$$

2.1 Family of λ -point *a*-ary approximating subdivision schemes

In this section, we are introducing family of λ -point *a*-ary approximating subdivision schemes for curve design for any integer; $a \geq 2$. Which is the extension of "B-spline". We have proved this family by using Chaikin [1], Hassan and Dodgson [11]. The Chaikin's algorithm for curve design is given by

$$\begin{cases} f^{k+1}_{2i} = \frac{3}{4}f^k_i + \frac{1}{4}f^k_{i+1}, \\ f^{k+1}_{2i+1} = \frac{1}{4}f^k_i + \frac{3}{4}f^k_{i+1}. \end{cases} \quad (2.9)$$

About twenty seven years later, it was extended to the 3-point scheme by Hassan and Dodgson and is given by

$$\begin{cases} f^{k+1}_{2i} = \frac{5}{16}f^k_{i-1} + \frac{10}{16}f^k_i + \frac{1}{16}f^k_{i+1}, \\ f^{k+1}_{2i+1} = \frac{1}{16}f^k_{i-1} + \frac{10}{16}f^k_i + \frac{5}{16}f^k_{i+1}. \end{cases} \quad (2.10)$$

The Laurent polynomials of (2.9) and (2.10) are

$$\begin{cases} P^2_2(z) = \frac{1}{4}\left(\frac{1-z^2}{1-z}\right)^2 \sum_{i=0}^{1}\binom{1}{i}z^i \\ P^2_3(z) = \frac{1}{16}\left(\frac{1-z^2}{1-z}\right)^3 \sum_{i=0}^{2}\binom{2}{i}z^i \end{cases}$$

If "a" represents arity then by generalizing, we get

$$P^a_\lambda(z) = \frac{1}{(2a)^{\lambda-1}}\left(\frac{1-z^a}{1-z}\right)^\lambda \sum_{i=0}^{\lambda-1}\binom{\lambda-1}{i}z^i \quad (2.11)$$

where integers λ, $a \geq 2$. From the coefficients of Laurent polynomial (2.11), we get the mask α^a_λ of family of λ -point *a*-ary approximating subdivision schemes for curve design for any integer λ, $a \geq 2$.

***Remark* 2.1**

• For $\lambda = 2$, a = 2, 3, 4, 5, 6 in (2.11), we get the mask of the following 2-point binary, ternary, quaternary, quinary and hexnary schemes, respectively,

$$\begin{cases} \alpha^2_2 = \frac{1}{4}\{1,3,3,1\}, \\ \alpha^3_2 = \frac{1}{6}\{1,3,5,5,3,1\}, \\ \alpha^4_2 = \frac{1}{8}\{1,3,5,7,7,5,3,1\}, \\ \alpha^5_2 = \frac{1}{10}\{1,3,5,7,9,9,7,5,3,1\}, \\ \alpha^6_2 = \frac{1}{12}\{1,3,5,7,9,11,11,9,7,5,3,1\}. \end{cases} \quad (2.12)$$

- For $\lambda = 3$, $a = 2,3,4,5,6$ in (2.11) we get the mask of the following 3-point binary, ternary, quaternary, quinary and hexnary schemes, respectively,

$$\begin{cases} \alpha^2_3 = \frac{1}{16}\{1,5,10,10,5,1\}, \\ \alpha^3_3 = \frac{1}{36}\{1,5,13,22,26,22,13,5,1\}, \\ \alpha^4_3 = \frac{1}{64}\{1,5,13,25,38,46,46,38,25,13,5,1\}, \\ \alpha^5_3 = \frac{1}{100}\{1,5,13,25,41,58,70,74,70,58,41,25,13,5,1\}, \\ \alpha^6_5 = \frac{1}{144}\{1,5,13,25,41,61,82,98,106,106,98,82,61,41,25,13,5,1\}. \end{cases} \quad (2.13)$$

- For $\lambda = 4$, $a = 2,3,4,5,6$ in (2.11) we get the mask of the following 4-point binary, ternary, quaternary, quinary and hexnary schemes, respectively,

$$\begin{cases} \alpha_4^2 = \frac{1}{64}\{1,7,21,35,35,21,7,1\}, \\ \alpha_4^3 = \frac{1}{216}\{1,7,25,59,101,131,131,101,59,25,7,1\}, \\ \alpha_4^4 = \frac{1}{512}\{1,7,25,63,125,203,277,323,323,277,203,125,63,25,7,1\}, \\ \alpha_4^5 = \frac{1}{1000}\{1,7,25,63,129,227,349,475,581,643,643,581,475,349,227,129,63,25,7,1\}, \\ \alpha_4^6 = \frac{1}{1264}\{1,7,25,63,129,231,373,543,733,907,1045,1123,1123,1045,907,733,547,373,231,129,63,25,1\}. \end{cases} \tag{2.14}$$

By adjusting the shape parameter in eq (2.11), we get λ-point a-ary parametric approximating subdivision scheme

$$P_\lambda^a(z) = \frac{1}{(2a)^{\lambda-1}}\left(\frac{1-z^a}{1-z}\right)^\lambda \sum_{i=0}^{\lambda-1}\binom{\lambda-1}{i} u_i z^i, \tag{2.15}$$

and

$$\sum_{i=0}^{\lambda-1}\frac{a}{2^{\lambda-1}}\binom{\lambda-1}{i} u_i = a, \;\; u_j - u_{\lambda-1-j}, \; j = 0,1,....,\lambda-2 \tag{2.16}$$

From the coefficients of Laurent polynomial (2.15) and (2.16), we get the mask α_λ^a of family of λ-point a-ary parametric approximating subdivision schemes for curve design for any integer λ, $a \geq 2$.

***Remark* 2.2**

- For $\lambda = 2$, a = 2, 3, 4 in (2.15) and (2.16), we get the mask of following 2-point binary, ternary and quaternary schemes respectively.

$$\begin{cases} \alpha_2^2 = \frac{1}{4}\{u_0, 3u_0, 3u_0, u_0\}, \\ \alpha_2^3 = \frac{1}{6}\{u_0, 3u_0, 5u_0, 5u_0, 3u_0, u_0\}, \\ \alpha_2^4 = \frac{1}{8}\{u_0, 3u_0, 5u_0, 7u_0 7u_0, 5u_0 3u_0, u_0\}, \end{cases} \tag{2.17}$$

- For $\lambda = 3$, a = 2, 3, 4 in (2.15) and (2.16), we get the mask of following 3-point binary, ternary and quaternary schemes respectively

$$\begin{cases} \alpha_3^2 = \frac{1}{16}\{u_0, 4+u_0, 12-2u_0, 12u_0-2u_0, 4+u_0, u_0\}, \\ \alpha_3^3 = \frac{1}{36}\{u_0, 4+u_0, 12+u_0, 24-2u_0, 28-2u_0, 24-2u_0, 12+u_0, 4+u_0, u_0\}, \\ \alpha_3^4 = \frac{1}{64}\begin{Bmatrix} u_0, 4+u_0, 12+u_0, 24+u_0, 40-2u_0, \\ 48-2u_0, 48-2u_0, 40-2u_0, 24+u_0, 12+u_0, 4+u_0, u_0 \end{Bmatrix} \end{cases} \tag{2.18}$$

- For $\lambda = 4$, a = 2, 3, 4 in (2.15) and (2.16), we get the mask of the following 4-point binary, ternary and quaternary schemes respectively,

$$\begin{cases} \alpha_4^2 = \frac{1}{64}\{u_0, 4+3u_0, 20+u_0, 40-5u_0, 20+u_0, 4+3u_0, u_0\}, \\ \alpha_4^3 = \frac{1}{216}\{u_0, 4+3u_0, 20+5u_0, 56+3u_0, 104-3u_0, 140-9u_0, 140-u_0\}, \\ \alpha_4^4 = \frac{1}{512}\begin{Bmatrix} u_0, 4+3u_0, 20+5u_0, 56+7u_0, 120+5u_0, 204-u_0, 284-7u_0, \\ 336-13u_0, 284-7u_0, 204-u_0, 120+5u_0, 56+7u_0, 20+5u_0, \\ 4+3u_0, u_0 \end{Bmatrix} \end{cases} \tag{2.19}$$

- For λ =5, a =2, 3, 4 in (2.15) and (2.16), we get the mask of the following 4-points binary, ternary and quaternary schemes respectively,

$$\begin{cases} \alpha_5^2 = \frac{1}{256}\begin{Bmatrix} u_0, 4+5u_0, 28+8u_0, 84, 140-14u_0, \\ 140-14u_0 84, 28+8u_0, 4+5u_0, u_0 \end{Bmatrix} \\ \alpha_5^3 = \frac{1}{1296}\begin{Bmatrix} u_0, 4+5u_0, 28+13u_0, 104+20u_0, 260+16u_0, 480-4u_0 \\ 684-30u_0, 768-42u_0, 684-30u_0, 480-4u_0, 260-+16u_0, \\ 104+20u_0, 28+13u_0, 4+5u_0, u_0 \end{Bmatrix} \\ \alpha_5^4 = \frac{1}{4096}\begin{Bmatrix} u_0, 4+5u_0, 104+25u_0, 280+36u_0, 600+36u_0, 1064+20u_0 \\ 1608-12u_0, 2014-50u_0 2400-74u_0, 2400-74u_0, 2014-50u_0, \\ 1608-12u_0, 1064+20u_0, 600+36u_0, 280+36u_0, 104+25u_0, \\ 28+13u_0, 4+5u_0, u_0 \end{Bmatrix} \end{cases} \tag{2.20}$$

For λ = 6, a = 2, 3, 4 in (2.15) and (2.16), we get the mask of the following 6-point binary, ternary and quaternary schemes respectively

$$\begin{cases} \alpha_6^2 = \frac{1}{1024}\begin{Bmatrix} u_0, 4+7u_0, 36+19u_0, 144+21u_0, 366-6u_0, 504-42u_0, \\ 504-42u_0, 336-6u_0, 144+21u_0, 36+19u_0, 4+7u_0, u_0 \end{Bmatrix} \\ \alpha_5^3 = \frac{1}{1296}\begin{Bmatrix} u_0, 4+7u_0, 36+25u_0, 168+57u_0, 528+87u_0, 1236+u_0, 2268+14u_0, \\ 3356-94u_0, 4068-178u_0, 4068-174u_0, 3356-94u_0, 2268+14u_0 \\ 1236+81u_0, 528+87u_0, 168+57u_0, 36+25u_0, 4+7u_0, u_0 \end{Bmatrix} \\ \alpha_6^4 = \frac{1}{32768}\begin{Bmatrix} u_0, 4+7u_0, 36+25u_0, 168+63u_0, 552+123u_0, 1428+189u_0, 3060+227u_0 \\ 5600+197u_0, 8928+74u_0, 12552-122u_0, 15688+326u_0, 17520-458u_0 \\ 15688-326u_0, 12552-122u_0, 8928+74u_0, 5600+197u_0, 3060+227u_0, \\ 1428+189u_0, 552+123u_0, 168+63u_0, 36+25u_0, 4+7u_0, u_0 \end{Bmatrix} \end{cases} \tag{2.21}$$

Table1: Different results of binary schemes

Scheme	Continuity	Support	Error Bounds
2-point binary	**C^1**	**3**	**0.025000**
3-point binary	**C^3**	**5**	**0.075000**
4- point binary	**C^5**	**7**	**0.125000**
5- point binary	**C^7**	**9**	**0.175000**
6- point binary	**C9**	**11**	**0.225000**

3. RESULTS AND DISCUSSIONS

In this section, we compare the different properties of the existing schemes as well as the proposed λ-point a-ary schemes generated by explicit formulae (2.15) and (2.16).

Table 2: Different results of ternary schemes

Scheme	Highest continuity	Support size	Error bounds
2-point ternary	C^1	2.5	0.008333
3-point ternary	C^2	4.0	0.033333
4-point ternary	C^4	5.5	0.058333
5-point ternary	C^5	7.0	0.083333
6-point ternary	C^7	8.5	0.108333

Table 3: Different results of quaternary schemes

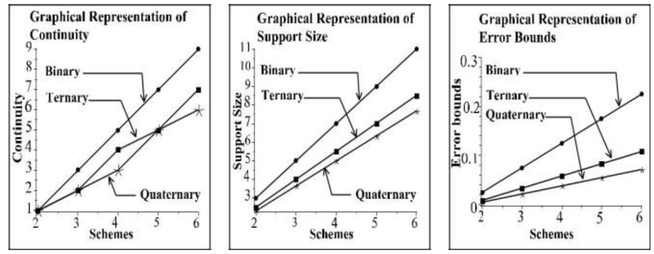
Scheme	Highest continuity	Support size	Error bounds
2-point quaternary	C^1	2.3333	0.004167
3-point quaternary	C^2	3.6667	0.020833
4-point quaternary	C^3	5.0000	0.037500
5-point quaternary	C^5	6.3333	0.054166
6-point quaternary	C^6	7.6667	0.070832

Table 4: Comparison: 2m-point and (2m + 1)-point a-ary interpolating schemes of Jian-ao-Lian [2]:

Schemes	C^n	Schemes	C^n
4-point binary	C^1	4-point ternary	C^1
6-point binary	C^2	6-point ternary	C^2
8-point binary	C^2	8-point ternary	C^2
3-point ternary	C^1	3-point quinary	C^0
5-point ternary	C^1	5-point quinary	C^0
7-point ternary	C^2	7-point quinary	C^1

Table 5: Comparison: (2b + 4)-point a-ary approximating and interpolating schemes of Mustafa and Najma [8]:

Approximating schemes	C^n	Interpolating schemes	C^n
4-point binary	C^5	4-point binary	C^1
6-point binary	C^5	6-point binary	C^2
4-point ternary	C^2	4-point ternary	C^2
6-point ternary	C^4	6-point ternary	C^2



(a) (b) (c)

Figure 1: (a), (b) and (c) represent the continuity, support size and error bounds of λ -point a-ary schemes, respectively.

In Table 1-3 and Fig. 1, we discussed the continuity, support size and error bounds of the generalized family of λ -point *a*-ary parametric approximating subdivision schemes (2.16). We note that continuity of the binary schemes is higher than ternary and quaternary schemes and it increases twice as the number of point increase by one. Continuity of the ternary schemes is greater than the continuity of the quaternary schemes. Here we see that support size and error bounds of the binary schemes are higher than ternary and quaternary schemes. It means like continuity, support size and error bounds of higher arity schemes generated by (2.16) are also less than the support size and error bounds of lower arity schemes. In Table 4, we calculated the continuity of already existing interpolating schemes introduced by Jian-ao-Lian [2]. Here we see that continuity of lower arity schemes is greater then higher arity schemes. In Table 5, we discussed the continuity of already existing approximating and interpolating schemes introduced by Mustafa and Najma [8]. Here we see that continuity of the binary schemes are higher than or equal to the ternary schemes. It is clear from Tables 1-5 that continuity of the proposed schemes is higher than existing schemes of [2, 8].

3.1 Special cases

1. Subdivision schemes generated by B-splines are special cases of our family of subdivision schemes (2.15). From the mask α_2^2, α_3^2 and α_4^2 which are defined by (2.17)-(2.21), we see that binary B-spline are also special cases of the schemes generated by (2.11).

2. By setting u_1 = 1/27, 1/72 and 1/72+ μ in (2.18), we get Hassan and Dodgson [11] 3-point ternary scheme, Siddiqi and Rehan 3-point ternary non-parametric and parametric schemes [16] respectively.

3. By setting $u_1 = 1/31104;\ u_2 = 76/31104$ in (2.19), we have mask of Siddiqi and Ahmad 5-point scheme [15].

By taking $\{a = 2;\ \lambda = 2\}$, $\{a = 3;\ \lambda = 3\}$ and $\{a = 4;\ \lambda = 4\}$ in the mask generated by (2.17), (2.18) and (2.19) we get Chaikin scheme [1], Hassan and Dodgson [11] and Ko [13] respectively.

3.2 CONCLUSION

We offered an explicit general formula, which generates the mask of all approximating subdivision schemes. We have also studied their continuity, support size, and obtained error bounds for them. It is observed that the continuity, support size and error bounds have increased by the increment in the complexity (number of point involved to insert new points) of the schemes while they have decreased by the increment in arity of the schemes. Moreover, schemes introduce by Chaikin [1], Hassan and Dodgson [11], Siddiqi and Rehan [15, 16] and Kowan [13] are special cases of our scheme. Continuity of proposed parametric schemes is better than the existing a-ary schemes [2, 8]. We concluded that by increasing arity, there is reduction in the continuity, support size, error bounds and computation cost of the subdivision schemes.

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